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Measuring Individual Vulnerability

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Abstract
Standard poverty analysis makes statements about deprivation after the veil of uncertainty has been lifted. This implies that there is no meaningful role for risk as part of an assessment of potentially low states of well-being. In this paper, we introduce a concept of vulnerability, as a threat of poverty, with downside risk at its core. More specifically, we define a vulnerability measure as an assessment of the magnitude of the threat of poverty, measured ex-ante, before uncertainty is resolved. We describe the welfare-economic foundations for desirable properties of a vulnerability measure and assess to what extent some measures used in empirical work abide by them. We also present two families of measures that are fully consistent with our axiomatic approach.

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> "About a quarter of the rural population survive by exchanging labour at market wages and commanding food with what they earn. For them a variation of the exchange relationships *can spell ruin*. There is, in fact, some evidence that in recent years in Bangladesh the wage system itself has moved more towards money wages, away from payments in kind –chiefly food. More modern, perhaps; *more vulnerable*, certainly" (Sen 1981, p. 150. The italics are ours).

1. Introduction

In recent decades, the welfare-economic foundations for the measurement of poverty have been clarified through seminal contributions by Sen (1976), Atkinson (1987) and many others. Poverty measures, consistent with social welfare functions satisfying reasonable axioms, such as the FGT-family of measures have found their way into applied research, using state-of-art methods and data analysis (e.g., Foster, Greer and Thorbecke, 1984; Ravallion, 1994).

Poverty measurement tends to involve three steps: the choice of a welfare indicator, the identification of the 'poor' via some norm, the poverty line, and an aggregation procedure. However, the entire analysis tends to take place in a world of certainty: poverty measures are defined after all uncertainty surrounding the individual welfare indicator has been resolved. In many instances this does not have to be a serious problem. For example, when assessing the impact of a new transfer scheme after it has been introduced, data on its actual impact and the resulting poverty outcomes are obviously relevant. However, when deciding to commit resources to competing schemes ex-ante, evaluating which one will be more effective to reduce poverty will have to take into account potential outcomes in different states of the world. Furthermore, the possibility of serious hardship contains information relevant for assessing low well-being. For example, consider two families, both with the same expected consumption, above some accepted norm, but one with a positive probability of hardship, and the other one facing no uncertainty. Neither is expected to be poor, and ex-post we may observe them to have the same consumption, but surely the possibility of downside risk for the former has some bearing on the ex-ante analysis of welfare.

It is surprising that the calculus of risk has not systematically entered normative economic analysis of poverty until fairly recently. Even Sen's (1981) seminal contribution on famines is in its welfare analysis concerned with the ex-post consequences of the crisis in terms of poverty and destitution. Policy analysis is done with the benefit of hindsight, even though the sequence of events unfolding during the Bangladesh famine in 1974 and the realised outcomes were just one set among a number of possible scenarios ex-ante.

In this paper we focus on the concept and measurement of vulnerability. We will use vulnerability as a measure of the threat of poverty. More specifically, vulnerability is used as the magnitude of the *threat* of poverty, measured ex-ante, before the veil of uncertainty has been lifted. This can be compared to poverty, which is itself the magnitude of low welfare outcomes, as observed without uncertainty and whereby low welfare is defined as outcome levels below some accepted poverty line.

Many authors have made use of the term 'vulnerability', with increased frequency since it was brought to the spotlight by the 2000/1 World Development Report, where "vulnerability measures the resilience against a shock – the likelihood that a shock will result in a decline in well-being" (World Bank 2001, p. 139). This definition may or may not appropriately fit the actual use of the term in particular papers within the vulnerability literature – in fact, there is no clear consensus as to how the term should be defined. However, no survey of the literature can fail to discover a common thread, which can probably be reduced to some sense of insecurity, of potential harm people must feel wary of – *something bad* can happen and 'spell ruin'.

Vulnerability is not the same as low expected welfare; neither is it merely tantamount for exposure to risk. In common parlance, someone is vulnerable if she is capable of being hurt or wounded. In fact, the etymological root 'vulnerare' is Latin for the verb 'to wound'. The term clearly relates to *dangers*, or *threats*, as opposed to uncertainties in general. For instance, in the example from Sen quoted above, we could say that, before the floods in Bangladesh in 1974, the future of wage-earners was overall more promising and less uncertain than that of subsistence farmers – yet their exposure to severe destitution in case of floods and food prices was greater, and they were more vulnerable than the farmers. There is a broader sense of the term 'vulnerability' as 'defencelessness', referring to a general frailty or helplessness of people. While our measure of vulnerability will include those who are bound to be poor in all states of the world, our focus in this paper is largely on exploring the implication of considering different possible states of the world, which both may or may not drive people into poverty.

This paper aims to contribute to the ongoing debate by proposing vulnerability measures which we will claim to be faithful to this fundamental sense of *vulnerability as exposure to 'threats', to 'downside risks'*. We will derive these measures from a set of axioms, including crucially what we will call the 'focus axiom'. This axiom will allow us to separate out threats from overall expectations, or in other words, downside risks from general uncertainty. Going back to Sen's quotation, even if Bangladeshi wage-earners expect a 'better future' than subsistence farmers, we will allow them to be more vulnerable than the latter.

There is a small, largely empirical but closely related literature that has introduced concepts of vulnerability (e.g. Bourguignon et al., 2004; Christiaensen and Subbarao, 2004; Chaudhuri et al., 2002; Kamanou and Morduch, 2004). Ligon and Schechter (2003) provide a conceptually careful attempt to bring poverty considerations into an expected utility framework for a well-defined concept of vulnerability. The analysis in this paper is fundamentally different by its normative welfare-economic focus, providing axiomatic foundations to measurement issues. It is non-welfarist in spirit, not relying on the utility framework. Furthermore, we place the notion of 'downside risk' at the core of our analysis.

The structure of the paper is as follows. Section 2 will discuss further our view of vulnerability, even though with no aim to be exhaustive. We intend to deal with some loose ends as we present and discuss the set of generic properties that we argue any measure of vulnerability should satisfy. Section 3 presents a first set of basic axioms, and ascertains whether existing measures abide by them. Section 4 proposes some additional properties which allow us to derive our two particular classes of measures. Section 5 concludes.

Finally, this paper is only concerned with individual vulnerability, and not with aggregation issues. A companion paper in progress will address this issue.

2. The concept of vulnerability

We view vulnerability as *the magnitude of the threat of future poverty*. This definition requires further clarification on a number points. First, we mean the ‘magnitude of the threat’ to relate both a) to the likelihood of suffering poverty in the future, and b) to the severity of poverty in such a case. Individuals dread the possibility of future poverty episodes, and they are said to be vulnerable to the extent that poverty cannot be ruled out as a possible scenario. By the same token, their vulnerability is greater when there is a worse danger to fear, when poverty threatens to be more severe.

Second, a threat remains as such until the uncertainty is resolved. The threat of an attack ceases when the enemy actually attacks, or when it becomes clear he will never do. Likewise, vulnerability is an ex-ante statement about future poverty, before the veil is lifted and the uncertainty is replaced by the knowledge of the actual facts.

Indeed, authors were prompted to resort to the term ‘vulnerability’ by the sense that the predicament of the poor is not only about insufficient command on resources, but also about insecurity and risks. The usual poverty concepts and measures do not capture the burden placed by this insecurity on the shoulders of the poor, as they typically focus on observed states of deprivation, making statements about singular or multiple dimensions of well-being. They invoke an ex-post concept of poverty, devoid of the ex-ante uncertainty which compounds the distress of the poor. In a sense, the notion of vulnerability was meant to amend this omission.

Third, we remark that, strictly speaking, we are referring to vulnerability *to poverty*. Individuals face several other threats such as illness, or crime, or loneliness. Yet we focus on the threat of poverty in particular, as this was also the focus other authors arguably had in mind when using the term 'vulnerability'. We thus understand expressions such as 'vulnerability to an epidemic' as a shortcut to 'vulnerability to poverty due to an epidemic'.

Of course, this choice opens up the question as to what concept of poverty we propose. We do not argue here in favour or against any particular view of the matter, and simply follow the mainstream by envisaging poverty as the failure to reach some minimum socially acceptable standard of living (as measured by overall consumption, or nutritional levels, or any other dimension of human well-being). We call this minimum standard the 'poverty line'.

Finally, we stress that, unless otherwise stated, we refer to *individual* vulnerability, as opposed to 'aggregate' vulnerability. Our unit of analysis is the individual agent, or the household. Given this unit, we do not intend to combine several units into one single vulnerability measure. We aim to assess how vulnerable each individual or household is, and not the extent of vulnerability among a group of them. We can thus rank families according to their vulnerability levels, or describe the evolution of household vulnerability over time – but we cannot compare vulnerability across regions or countries. A companion paper will address this issue, entirely ignored in the earlier more empirical contributions on this topic. It shows that aggregation faces a number of difficulties which must be carefully dealt with.

3. Basic properties of a vulnerability measure

3.1. A first set of axioms

We formulate our desirable properties of vulnerability measures as a set of axioms. While we avoid in this section mathematical proofs and aim to focus on the intuitions behind the axioms, we still need to present our preferred notation.

Let vulnerability be measured by

$$V^{*}=V(z,\mathbf{y},\mathbf{p}),$$

where z is the poverty line, **y** is a vector of outcomes across n states of the world, and a vector **p** of corresponding probabilities. It may be easiest to think of these outcomes as consumption levels, but we shall avoid such language as an effort to stress our measure is suitable to other well-being dimensions.

Note the elements of **y** and **p** are paired together, i.e. p_i is the probability of outcome y_i occurring. Domains are determined by

$$\mathbf{y}\in\left\{(y_1,y_2,...,y_n)\middle| y_i\in[0,\infty[\right\}\text{, and }\mathbf{p}\in\left\{(p_1,p_2,...,p_n)\middle| p_i\in[0,1]\wedge\sum_{i=1}^{n}p_i=1\right\}$$

The poverty line z distinguishes 'good' from 'bad' outcomes, so that the individual is poor in the i-th state of the world if $y_i<z$.

Note we assume that the poverty line is common to all states of the world. This may be unappealing if, for example, some states of the world (e.g. an earthquake) require outcomes to drastically improve in order to preserve an acceptable living standard. For simplicity, we ignore such refinement.

Having laid down our basic notation, we now turn to our first axiom:

AXIOM 1 – SYMMETRY. V* satisfies this axiom if for every (z,**y**,**p**) and any permutation mapping σ: {1, …, n}→{1, …, n},

$$V\left(z,(y_1,y_2,...,y_n),(p_1,p_2,...,p_n)\right)=V\left(z,(y_{\sigma(1)},y_{\sigma(2)},...,y_{\sigma(n)}),(p_{\sigma(1)},p_{\sigma(2)},...,p_{\sigma(n)})\right). \quad (A1)$$

This axiom ensures the measure is not sensitive to permutations of the states of the world, i.e. all states receive the same treatment. As far as vulnerability is concerned, the only relevant difference between two states of the world i and j is the difference in their outcomes y_i and y_j. All other features are uninteresting, and states of the world can swap 'labels' with no information loss – for instance, an illness and a bad harvest are equivalent if they occur with equal probability and have the same effect on our outcome at hand. Admittedly, it could be argued that individuals have more reasons to fear illnesses than a bad harvest, and that the latter is perceived as less of a threat. We ignore such distinctions in this paper.

We will invoke this axiom repeatedly from very early on, as we shall use it to simplify our formulation of some of the following axioms. Given symmetry, we can simply express them as properties of one particular state of the world, and then simply extend them to all other states without further ado. We will make no explicit recall of Axiom 1 in those cases.

The next axiom will require some additional notation. Define $\tilde{y}_i$=Min(y_i,z) and $\tilde{\mathbf{y}}=(\tilde{y}_1,\tilde{y}_2,\ldots,\tilde{y}_n)$, i.e. $\tilde{y}_i$ is a censored outcome measure, where outcomes beyond the poverty line are equated to the poverty line itself.

AXIOM 2 – FOCUS: V* satisfies this axiom if for every (z,**y**,**p**),

$$V(z,\mathbf{y},\mathbf{p}) = V(z,\tilde{\mathbf{y}},\mathbf{p})\,. \qquad \text{(A2)}$$

As long as outcomes are above the poverty line, our vulnerability measure will ignore changes in those outcomes – the uncensored vector **y** will not add any relevant contribution to the information already contained in $\tilde{\mathbf{y}}$. Greater or lower outcomes in ‘good’ states of the world do not make individuals more or less vulnerable to poverty.

This axiom is obviously related to our view of vulnerability as a burden caused by the threat of future poverty, as it ensures this burden will not be compensated by simultaneous (ex-ante) possibilities of being well-off. Good overall expectations are not the same as, nor do they ensure, low vulnerability. Take again the example of subsistence farmers and wage-earners in Bangladesh: the fact that the latter would have probably thrived and outperformed the former in almost every other state of the world apart from the floods does not in any way mitigate their vulnerability.

Take also the following example from the Sahel: “Compared with the farmer or the pastoralist who lives on what he grows and is thus vulnerable only to variations of his own output (arising from climatic considerations or other influences), the grower of cash crops, or the pastoralist heavily dependent on selling animal products, is vulnerable both to output fluctuations and to shifts in marketability of commodities and in exchange rates. (…) *while commercialization may have opened up new* *economic opportunities, it has also tended to increase the vulnerability of the Sahel population*" (Sen, 1981, p. 126. The italics are ours).

AXIOM 3 – PROBABILITY-DEPENDENT EFFECT OF OUTCOMES. V* satisfies this axiom if for every (z,**y**,**y**′,**p**,**p**′,**p**″), with $y_1<z$, and $p_1=p_1'\neq p_1''$, and $d>0$,

$$\begin{aligned}&V\big(z,(y_1,y_2,...,y_n),(p_1,p_2,...p_n)\big)-V\big(z,(y_1+d,y_2,...,y_n),(p_1,p_2,...p_n)\big)\\&=V\big(z,(y_1,y_2',...,y_n'),(p_1,p_2',...p_n')\big)-V\big(z,(y_1+d,y_2',...,y_n'),(p_1,p_2',...p_n')\big)\end{aligned}\qquad\text{(A3a)}$$

and

$$\begin{aligned}&V\big(z,(y_1,y_2,...,y_n),(p_1,p_2,...p_n)\big)-V\big(z,(y_1+d,y_2,...,y_n),(p_1,p_2,...p_n)\big)\\&\neq V\big(z,(y_1,y_2,...,y_n),(p_1'',p_2'',...p_n'')\big)-V\big(z,(y_1+d,y_2,...,y_n),(p_1'',p_2'',...p_n'')\big)\end{aligned}\qquad\text{(A3b)}$$

Should the outcome in one state of the world improve, the consequent effect on vulnerability is not allowed to depend on the outcomes or probabilities of other states of the world (as imposed by the equality above), but it must be sensitive to the likelihood of that particular state of the world (as imposed by the inequality). We discuss either part of this axiom in turn.

The first part might not look intuitively appealing at first glance, and yet an example can cogently make the case for it. Consider the case of the cash-crop farmer in Sahel, and imagine only two states of the world are possible, drought and rain. Output is higher in the latter case, but even then it fails to reach the poverty line. Now if output in the case of drought improves (say due to new technology), the farmer might now be more or less (the next axiom will impose she is actually less) vulnerable than before. Should the *magnitude* of the effect on vulnerability depend on the outcome with rain?

Axiom 3 answers in the negative, for the simple reason that, given a drought, then the severity of the farmer's poverty cannot be in any way relieved by the thought of the outcome she would have had, had the weather gods been more generous. Vulnerability focuses on how threatening poverty episodes are, because there is in fact a danger of suffering poverty. What ultimately matters is the sense of deprivation the individual will actually feel in the realised state of the world, and this is the threat we are concerned with.

We realise that a possible counterargument could run 'in fact, there could be some relief in considering that one could have done much better had the odds been more fortunate' (or to the contrary, 'she may rue having missed a better outcome, with no fault on her part, and thus her misery will be greater'). There is some truth in these arguments, and yet we dismiss them as we search for a measure based on 'objective' (albeit ex-ante, potential) poverty – as we said, we simply adhere to the common concept of poverty as failure to reach a poverty line, and we consider this is the path an measure of 'objective' (as opposed to 'subjective') vulnerability needs to follow.

The intuition behind the second part of the axiom is clearer. Say vulnerability lessens after the improvement in outcomes under drought, should we expect this effect on vulnerability to be the same when a drought is a very unlikely event and when it is a clear and present danger? Arguably not. This is the answer imposed by Axiom 3.

AXIOM 4 – PROBABILITY TRANSFER. V* satisfies this axiom if for every (z,**y**,**p**) and $p_1 \geq e > 0$,

$$V\left(z,\mathbf{y},(p_1,p_2,\ldots p_n)\right)\begin{Bmatrix}\geq\\ \leq\end{Bmatrix}V\left(z,\mathbf{y},(p_1-e,p_2+e,\ldots,p_n)\right) \text{ if and only if } \tilde{y}_1\begin{Bmatrix}\leq\\ \geq\end{Bmatrix}\tilde{y}_2. \quad \text{(A4)}$$

If y_i is less than or at most equal to y_j, then vulnerability cannot increase as a result of a probability transfer from state i to state j. Likewise, if y_i is greater than or at least equal to y_j, then vulnerability cannot decrease. The use of censored outcomes $\tilde{y}_{i,j}$ in the right-hand inequalities ensures that changes above the poverty line are ignored.

The intuition behind this axiom is clear enough. The Sahelian farmer surely becomes more vulnerable if a drought becomes more likely, at the expense of the rainy scenario. However, we need to remark at least two implications of this statement. First, note the effect of the probability transfer is not allowed to depend on the initial probability distribution. To put it differently, our vulnerability measures will be linear in probabilities.

Again, a counterargument can follow and claim that a change in probability should have a stronger effect when the event is initially very (un)likely. However, we prefer to disregard this claim, and thus we steer clear of the danger of some counterintuitive results, as for instance the possibility that, under some probability distributions, a probability transfer from rain to drought could make the farmer *less* vulnerable.

Secondly, Axiom 4 implies that increases in vulnerability are monotonically related to decreases in outcomes (as long as outcomes are below the poverty line). Such drops in outcomes mean individuals are vulnerable to greater poverty. By the same token, greater probability of a low-outcome state means greater vulnerability. Clearly enough, this condition is closely linked to our view of vulnerability as the threat of poverty, which is more dreadful when looming poverty episodes are more severe.

AXIOM 5 – RISK SENSITIVITY. V* satisfies this axiom if for every (z,**y**,**p**),

$$V(z,\mathbf{y},\mathbf{p}) > V(z,(\hat{y},\hat{y},...,\hat{y}),\mathbf{p}), \text{ where } \hat{y} = \sum_{i=1}^{n} p_i \tilde{y}_i . \qquad \text{(A5)}$$

Vulnerability would be lower if the expected (censored) outcome ŷ were attained with certainty, i.e. if with no need to increase outcome expectations, uncertainty were removed by making the final outcome independent of the state of the world realisation. Put it differently, the existence of risk leads to greater vulnerability.

Note we take the expectation of censored outcomes here, because $\tilde{\mathbf{y}}$ conveniently neglects the 'excess' outcome whenever y_i>z. Loosely speaking, the excess over z is a wasteful outcome, as far as vulnerability is concerned. Loosely speaking again, $\tilde{\mathbf{y}}$ denotes an 'effective' outcome, and so does ŷ.

This implies that we implicitly define an increase in risk as a probability transfer 'from the middle to the tails', in keeping with one of the Rothschild-Stiglitz senses of risk. Indeed, the right-hand side in (A5) assumes ŷ occurs with certainty – the probabilistic weight falls entirely on ŷ. The left-hand side spreads that weight away from the expected outcome, towards the tails, and risk and vulnerability are consequently greater.

Needless to say, accepting this definition of risk immediately leads to a statement such as Axiom 5, where vulnerability is ensured to increase in risk. Axiom 5 links up with our first intuition about vulnerability, as a concept aiming to capture the burden of insecurity, the fact that hardship is also related to fear for the future, to threats. The cash-crop farmer in Sahel must be more vulnerable to poverty if output prices are more variable (say due to more limited connections to the market) with no increase in their expected value.

For later use, we now turn to an alternative interpretation of Axiom 5, building on the notion of the certainty-equivalent outcome $y^C = y^C(z,\mathbf{y},\mathbf{p})$, which is defined by

$$V(z,\mathbf{y},\mathbf{p}) = V\left(z,(y^C, y^C,..., y^C),\mathbf{p}\right), \text{ for any } (z,\mathbf{y},\mathbf{p}).$$

When all states of the world yield the same outcome y^C, so that no uncertainty exists, the individual is as vulnerable as in the original, uncertain scenario ($\mathbf{y}$,$\mathbf{p}$). Note there is an inverse relation between V* and y^C – given that Axiom 4 imposes vulnerability shall monotonically decrease in outcomes, a rise in y^C will prompt a decrease in V*. From the monotonicity result,

$$V\left(z,(y^C, y^C,..., y^C),\mathbf{p}\right) > V\left(z,(\hat{y}, \hat{y},..., \hat{y}),\mathbf{p}\right) \text{ implies } y^C<\hat{y},$$

where the first inequality is just another way to write (A5). Loosely speaking, Axiom 5 signifies an 'efficiency loss' in the distribution of outcomes across states of the world. Intuitively, imagine some form of insurance becomes available, such that uncertainty is removed and ŷ becomes the actual outcome level – the individual would then be a position to give up $\hat{y}$-y^C outcome units and remain as vulnerable as at the outset. In other words, the existence of uncertainty means that outcomes will to some extent fail to translate into low vulnerability.

Finally, risk sensitivity will be crucial to show that our vulnerability measures must be convex in the outcome of any particular state of the world (again, as long as $y_i<z$). Along with convexity, it imposes continuity on the reaction to changes in such outcomes.

AXIOM 6 – SCALE INVARIANCE. V* satisfies this axiom if for every (z,**y**,**p**) and λ>0,

$$V(z,\mathbf{y},\mathbf{p}) = V(\lambda z,\lambda\mathbf{y},\mathbf{p}) \qquad \text{(A6).}$$

Equal proportional changes in the poverty line (z) and outcomes (y_i) make the individual neither more nor less vulnerable. This axiom conveniently allows our measure not to depend on the unit of measure of outcomes, at the cost of imposing that relative distance from the poverty line is all that matters.

Scale invariance closes our set of basic axioms. In fact, axioms 1 to 6 turn out to confine the analysis to the class of measures where vulnerability is a probability-weighted average of state-specific 'deprivation indices', which we define below. Theorem 1 formalises this result.

THEOREM 1 – If V* satisfies Axioms 1-6, then

$V^* = \sum_{i=1}^{n} p_i v(x_i)$, where $x_i = \frac{\tilde{y}_i}{z}$ and v(.) is monotonically decreasing and convex.

By 'deprivation index' we mean a monotonic, convex transformation of the rate of coverage of the minimal needs. This rate is measured by x_i and necessarily lies in the [0,1] interval. Needless to say, it reaches 1 when the outcome equals or exceeds the poverty line. In such case, we say that the individual has met her minimal needs, i.e.

those needs society regards as basic. When $x_i<1$, the individual is poor, and her hardship worsens as x_i decreases further away from 1.

When x_i is transformed by a monotonically decreasing, convex function, the result $v(x_i)$ is meant to measure state-specific deprivation. While the inverse relation between x_i and $v(x_i)$ needs no explanation, the convexity requirement is not self-evident. The proof in the Appendix shows that it follows from our risk-sensitivity axiom – the effect on vulnerability of a fall in the outcome of a particular state must decrease in the initial level of that outcome.

3.2. Comparison with other measures

In this section we ascertain whether current vulnerability measures satisfy our set of basic desiderata. For convenience, we distinguish two groups of measures. The first group builds on explicitly welfarist grounds and envisages vulnerability as low expected utility. The second group is more popular and focuses on expected poverty, as measured by the usual axiomatic indices.

We overlook here studies where vulnerability is understood as inability to isolate well-being from income shocks, e.g. as in Amin, Rai and Topa (2003). For instance, in a regression of consumption on income and other variables, the income coefficient would be a construed as a measure vulnerability.

In such view, outcome changes are all that matters – outcome levels are irrelevant, as well as the concept of a critical outcome level (as the poverty line). The threshold to define a bad outcome is either initial consumption, or expected consumption (which is the implicit assumption of regression-based analyses). Furthermore, the probabilities of shocks occurring play no role – these measures focus on reaction to the shock, given the shock occurs. To put it briefly, this approach to vulnerability is so far removed from our framework, that in fact any comparison with the axioms above would be a meaningless exercise.

Welfarist measures

A few studies have based their analyses on explicit welfare foundations (e.g., Cunningham and Maloney, 2000; Ligon and Schecher, 2003; Elbers and Gunning, 2003). Both Ligon and Schechter, and Elbers and Gunning take a utilitarian stance and view vulnerability as *'low' expected utility*, where 'low' can be further specified by defining some minimum socially acceptable utility level.

With our notation, Ligon and Schechter's measure can be written as

$$V^{LS} = \sum_{i=1}^{n} p_i \left[U(z) - U(y_i) \right]$$, where U(.) is a well-behaved utility function.

Provided utility exhibits the usual properties (e.g. continuity, monotonicity, concavity), all our axioms are satisfied, with two exceptions. Firstly, scale invariance is not necessary. Were we to impose it on V^{LS}, so as to ensure changes in measurement units are meaningless, then the set of acceptable utility functions would dramatically narrow down to only one choice, namely a logarithmic form, such that

$$V^{LS} = \sum_{i=1}^{n} p_i \ln\left(\frac{z}{y_i}\right)$$

This form is not without drawbacks. For instance, it is not defined for cases where the outcome falls to zero in some particular state of the world.

More crucially, these measures are bound to violate the focus axiom, at least as they have been formulated to date. If vulnerability depends on expected utility in general, it will be necessarily sensitive to the likelihood and the magnitude of 'good' outcomes. Even if severe destitution is one possible scenario, a household need not be seen as vulnerable, provided other scenarios are promising enough to compensate for the fear of starvation. In other words, these measures could have classified our Bangladeshi wage-earners as non-vulnerable.

Ignoring the focus axiom also leads to some odd conclusions, as two examples will clarify further. Firstly, let us imagine that the poor buy each week a state lottery ticket – they spend a very small sum of money, but 'you never know', and there is a 0.001 percent chance of winning to the top prize of $10,000. The following 'policy' measure would make these households less vulnerable, as measured by V^{LS}: increase the top prize to $10 million!

For a second example, assume rain and drought are the only two states of the world possible, and the povery line is estimated to be 100. Imagine V^{LS} finds John (with outcomes (80,50), under rain and drought respectively) more vulnerable than his neighbour Peter (with outcomes (120,30)).

Imagine now that the poverty line had been overestimated (say because the researcher wrongly thought that John and Peter had special needs). If the real poverty line is 70 and V^{LS} is recalculated, should we still expect John to turn out more vulnerable than Peter? By construction, V^{LS} rules out any change in the vulnerability ranking, and hence it is bound to answer in the affirmative.[1]

We find however no strong intuition to a priori discard any ranking reshuffle. In fact, when the true poverty line is used, John's future has some scope for hope – should the weather be benevolent and the rain plentiful, he would escape poverty, along with Peter. Nevertheless, if the line is overestimated, John is doomed to destitution, even in the best scenario, whereas Peter's hopes remain upbeat. If we take the poverty line seriously, then this should have relevant consequences in our assessment of vulnerability. V^{LS} overlooks such consequences, and the reason lies in the peripheral role of the poverty line. In the case of V*, this line is placed at the core of the analysis, by virtue of the focus axiom. Its changes will thus be allowed to alter vulnerability rankings, as we shall verify later.

Expected-poverty measures

Measures in this group were inspired by Ravallion (1988). Christiaensen and Subbarao (2004), Suryahadi and Sumarto (2003), Kamanou and Morduch (2004), and Chaudhuri, et al. (2002) are recent examples. They all see vulnerability as expected poverty, and differ only in the time spells they consider, or in the econometric strategies they deploy.

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[1] As an example, take $U(y)=y^{0.1}$, and let the probability of rain be 90%. Then V_{JOHN}=0.042 and V_{PETER}=-0.008 when the poverty line was believed to be 100, but V_{JOHN}=-0.013 and V_{PETER}=-0.064 when it is revised downwards. In both cases, $V_{JOHN}>V_{PETER}$. The ranking is unaltered by the change in the poverty line.

As poverty is usually measured by FGT indices (Foster et al., 1984), here we may write vulnerability (V^{EP}) as

$$V^{EP} = \sum_{i \forall y_i < z} p_i \left(\frac{z - y_i}{z} \right)^a \text{ where } a \geq 0,$$

which clearly satisfies all our axioms thus far, except for Axioms 4 and 5, which hinge on the choice of a.

Since this approach inherits the features of the poverty measure at hand, it does not come as a surprise to find that the focus axiom applies. Likewise, the well-known drawbacks of a=0 and a=1 carry over to this vulnerability measure. They remain important caveats, all the more because the empirical literature resorts to both the probability of being poor (a=0) and the expected shortfall (a=1) with great frequency.

In terms of our desiderata, a=0 fails to meet Axiom 4, and more interestingly a=1 is at odds with Axiom 5.[2] Ligon and Schechter (2003) were the first to point out the shortcomings of V^{EP} with regard to the welfare burden due to risk exposure. Just as the poverty gap is insensitive to the distribution of outcomes among the poor, a=1 implies that the vulnerability measure will pay no attention to the probability distribution of outcomes below the poverty line. In other words, it assumes risk-neutrality.

[2] The probability of being poor (a=0) is not sensitive to a probability transfer from state i to state j, with $y_i < y_j < z$.

Ligon and Schechter explained further that $a \neq 1$ might not suffice as a corrective. For instance, $0<a<1$ implies that greater risk will mitigate vulnerability (and hence Axiom 5 would again remain unfulfilled). Moreover, even though $a>1$ would secure all our axioms, it also proves to be a troublesome condition, as it imposes that better outcomes will exacerbate the extent to which the individual dreads an increase in risk exposure, in spite of empirical evidence to the contrary. The next section discusses this point in greater detail.

4. Two classes of individual vulnerability measures

4.1. Normalisation and more on risk sensitivity

We now turn to some additional axioms which will further narrow down the set of acceptable measures. Admittedly, both axioms will be intuitively appealing, and yet not compelling. Their major advantage will lie in their power to simplify the analysis.

AXIOM 7 – NORMALISATION. V* satisfies this axiom if for every (z,**p**),

$$\max_{y}\left\{V\left(z,\mathbf{y},\mathbf{p}\right)\right\}=1, \text{ and } \min_{y}\left\{V\left(z,\mathbf{y},\mathbf{p}\right)\right\}=0 \qquad \text{(A7).}$$

Given the poverty line and a probability distribution **p**, state-specific outcomes **y** can vary and alter the level of vulnerability, but this level will be bounded by the [0,1] interval. This axiom greatly simplifies the intuitive interpretation of the measure, and needs no further defence.

AXIOM 8 – CONSTANT RELATIVE RISK SENSITIVITY. V* satisfies this axiom if for every (z,**y**,**p**),

$$y^C(z,\mathbf{y},\mathbf{p}) = \frac{y^C(z,\kappa\mathbf{y},\mathbf{p})}{\kappa}, \text{ where } \kappa>0 \qquad \text{(A8).}$$

A *proportional* increase by κ in the outcomes of all possible states of the world leads to a similar *proportional* increase in the certainty-equivalent outcome y^C. With other words, y^C is homogenous of degree one. While Axiom 5 ensures $y^C/\hat{y}<1$, Axiom 8 further imposes that this ratio shall remain constant if all state-specific outcomes increase proportionally, i.e. the 'efficiency loss' due to uncertainty is determined as a constant proportion of expected outcome.

Needless to say, this axiom cannot be imposed as forcefully as any of those in our basic set. For instance, one might alternatively prefer to propose that the *absolute* increase in the certainty-equivalent outcome equates the *absolute* increase in state-specific outcomes, i.e. the 'efficiency loss' is a constant value y^C-ŷ. Axiom 9 follows this path. Of course, risk sensitivity could be assumed to behave otherwise, probably on the grounds of empirical investigations into perceptions of risk and attitudes towards it. For the purpose of this paper, it will be enough to note that each of these choices will call for a specific axiom and a specific class of measures. We contemplate here only two cases, as defined by Axioms 8 and 9.

AXIOM 9 – CONSTANT ABSOLUTE RISK SENSITIVITY. V* satisfies this axiom if for every (z,**y**,**p**),

$$y^C(z,\mathbf{y},\mathbf{p})+\tau = y^C(z,\mathbf{y}',\mathbf{p})\text{, where } \tau>0 \text{ and } y'_i=y_i+\tau \qquad \text{(A9).}$$

Theorems 2 and 3 propose two classes of vulnerability measures, each related to a particular assumption about risk sensitivity patterns. Their proofs are postponed to the Appendix. To begin with, we first assume constant relative risk sensitivity.

THEOREM 2 – If V* satisfies AXIOMS 1-8, then

$$V^*_\alpha = 1 - \mathrm{E}\left[x_i^\alpha\right]\text{, where } 0<\alpha<1.$$

We highlight the simplicity of this single-parameter familiy of measures V^*_α. We find a less attractive class under constant absolute risk sensitivity (V^*_β), as shown by Theorem 3.

THEOREM 3 – If V* satisfies AXIOMS 1-7 and 9, then

$$V^*_\beta = \mathrm{E}\left[\frac{e^{\beta(1-x_i)}-1}{e^\beta - 1}\right]\text{, where } \beta>0.$$

To interpret this measure, note the denominator takes the same form as the numerator, except it assumes x_i=0. In other words, state-specific deprivation indices compare in some way actual outcomes with the worst possible scenario, where the outcome is allowed to fall to zero.

4.2. Some convenient features of these classes of vulnerability measures

We discuss here some properties of *both* V^*_α and V^*_β. We highlight four main features:

PROPERTY 1 – *People who are certain to be poor are highly vulnerable*. This goes back once again to our view of vulnerability. Indeed, if vulnerability is about threats, certainty of being poor is but a dominant, irresistible threat. The concept is not restricted to those whom the winds might blow into poverty or out from it.

PROPERTY 2 – V^* is equal to the *probability of being poor* (π) only if consumption is bound to be zero in every state of the world where the individual is poor.[3] In most cases, V^* will be lower than this probability.

PROPERTY 3 – As expected, *the vulnerability ranking generated by V* is sensitive to changes in the poverty line z*. In the comparison between John and his neighbour Peter, both V^*_α and V^*_β allow ranking reshufflings to occur after the poverty line is revised. [4]

PROPERTY 4 – Both α and β can be construed as determining the *degree of risk sensitivity*. In particular, increasing risk exposure will be related to a greater increase in measured vulnerability when α is high, or when β is low.[5] This relation can in fact guide the choice of values for these parameters in empirical applications.

[3] To prove it, simply say $x_i=0$ if $i \le n^* \le n$ and $x_i=1$ if $i>n^*$. $V^*_\alpha=V^*_\beta=\Sigma_{i\le n^*}p_i=\pi$ will follow immediately.

[4] Taking the numbers of note 1, and $\alpha=0.1$ for the sake of the comparison, $V_{JOHN}>V_{PETER}$ when the poverty line is 100 ($V_{JOHN}=0.027$ and $V_{PETER}=0.011$), and $V_{JOHN}<V_{PETER}$ when it is revised downwards ($V_{JOHN}=0.003$ and $V_{PETER}=0.008$). The ranking does change.

[5] The certainty-equivalent outcome proves to be a simple tool to ascertain the role of α and β. Considering the top boundary for α, we find $y^C=\hat{y}$, i.e. uncertainty is meaningless and causes no

5. Concluding remarks

Standard poverty analysis makes statements about deprivation after the veil of uncertainty has been lifted. This implies that there is no meaningful role for risk as part of an assessment of low states of well-being; the only role is instrumental. In this paper, we introduced a concept of vulnerability, as a threat of poverty. We used a set of axioms for desirable properties for a welfare-economic assessment of vulnerability. In practice, these properties combined well-known poverty axioms with postulates about the effects of varying exposure to risk.

In recent years, quite a few empirical papers have been produced using some concept of vulnerability not dissimilar to our own. However, we can show that they all fail to satisfy our axioms fully. In a series of theorems, we present vulnerability measures that abide by them.

The contribution in this paper has been to highlight the welfare-economic axiomatic foundations of a vulnerability measure at the individual level. The issue of aggregation of a vulnerability measure requires more careful consideration, well beyond the haphazard approach currently used in many of the empirical contributions. This will be discussed in a subsequent paper. Furthermore, there are crucial challenges in making these concepts operational, although the many interesting applications already circulating in the literature and referred to before show the potential. More applications are being prepared at present by the authors, directly linked to the approach described in this paper.

'efficiency loss'. At the other end of the interval ($\alpha \rightarrow 0$), $\ln(y^C)=\Sigma_i p_i \ln(y_i)<\ln(\hat{y})$, so that Jensen's inequality ensures $y^C<\hat{y}$. In the case of β, $y^C=\hat{y}$ results from taking the bottom boundary ($\beta \rightarrow 0$). When $\beta \rightarrow \infty$, $y^C=\mathrm{Min}_i(y_i)<\hat{y}$.

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Appendix

Proof of Theorem 1

Our starting point is a self-explanatory corollary from Axiom 4.

COROLLARY 1 – If V* satisfies Axiom 4, then for every (z,**y**,**p**) with $\tilde{y}_1=\tilde{y}_2$, and $p_1\geq e>0$, then $V(z,\mathbf{y},(p_1,p_2,\ldots,p_n))=V(z,\mathbf{y},(p_1-e,p_2+e,\ldots,p_n))$.

Next we use Axioms 1 to 3. The first equality below follows from (A3), while the second one removes the subscript from function ψ^* and is a direct application of (A1) and (A2), as **y** is replaced by $\tilde{\mathbf{y}}$.

$$V(z,\mathbf{y},\mathbf{p})=\theta(z,\mathbf{p})+\sum_{i=1}^{n}\psi_i^*(z,y_i,p_i)=\theta(z,\mathbf{p})+\sum_{i=1}^{n}\psi^*(z,\tilde{y}_i,p_i).$$

From (A1), function θ must be symmetric. Let **p′** denote the probability vector after a probability transfer from p_1 to p_2, and note that Corollary 1 can now be written as

$$\sum_{i=1}^{2}\psi^*(z,y_i,p_i)-\sum_{i=1}^{2}\psi^*(z,\tilde{y}_i,p_i')\begin{Bmatrix}\geq\\ \leq\end{Bmatrix}\theta(z,\mathbf{p}')-\theta(z,\mathbf{p}) \text{ iff } \tilde{y}_1\begin{Bmatrix}\leq\\ \geq\end{Bmatrix}\tilde{y}_2$$

As elements of **p** other than p_1 and p_2 are not allowed alter these inequalities, transfers among them have no effect on θ(z,**p′**)-θ(z,**p**). Given symmetry, the same applies to p_1 and p_2, and hence

$$k_1=\theta(z,(p_1+\delta_1,p_2+\delta_2,\ldots,p_n+\delta_n))-\theta(z,(p_1,p_2,\ldots,p_n)), \text{ where } \sum_{i=1}^{n}\delta_i=0$$

and k_1 is a constant. Let δ_i=1-p_i and rearrange so that

$$\theta(z,(p_1,p_2,\ldots,p_n))=\theta(z,(1,1,\ldots,1))-k_1=\theta^*(z).$$

We then define

$$\psi(z,\tilde{y}_i,p_i)=\frac{\theta^*(z)}{n}+\psi^*(z,\tilde{y}_i,p_i)$$

so that

$$V(z,\mathbf{y},\mathbf{p})=\sum_{i=1}^{n}\psi(z,\tilde{y}_i,p_i) \qquad \text{(R1).}$$

The effect of each state of the world on total vulnerability can be caught by a function ψ of its censored outcome and its probability. Total vulnerability is made up by these state-specific contributions.

Next we find our measure is linear in probabilities. From Corollary 1 and (R1), if $\tilde{y}_1 = \tilde{y}_2$ and $p_1>e>0$,

$$\begin{aligned}&\psi(z,\tilde{y}_1,p_1)+\psi(z,\tilde{y}_1,p_2)+\sum_{i=3}^{n}\psi(z,\tilde{y}_i,p_i)\\ &=\psi(z,\tilde{y}_1,p_1-e)+\psi(z,\tilde{y}_1,p_2+e)+\sum_{i=3}^{n}\psi(z,\tilde{y}_i,p_i)\end{aligned}$$

Defining $p_1'=p_2+e$ and rearranging,

$$\frac{\psi(z,\tilde{y}_1,p_1)-\psi(z,\tilde{y}_1,p_1-e)}{p_1-(p_1-e)}=\frac{\psi(z,\tilde{y}_1,p_1')-\psi(z,\tilde{y}_1,p_1'-e)}{p_1'-(p_1'-e)},$$

which implies the absolute rate of change is independent of p_1. In general, this implies ψ must be linear in p_i:

$$\psi(z,\tilde{y}_i,p_i)=\upsilon_0(z)+p_i\upsilon_1(z,\tilde{y}_i),$$

where (A3) precludes υ_0 from depending on $\tilde{y}_i$. Define

$$\upsilon(z,\tilde{y}_i)=\upsilon_0(z)+\upsilon_1(z,\tilde{y}_i).$$

Using (R1) again we have, for $p_1\geq e>0$,

$$V(z,\mathbf{y},\mathbf{p})=\sum_{i=1}^{n}p_i\upsilon(z,\tilde{y}_i) \qquad \text{(R2).}$$

Next we show that function $\upsilon(z,\tilde{y}_i)$ must monotonically decrease in $\tilde{y}_i$. If outcome in some state of the world is below the poverty line, then an increase in this outcome causes a decrease in vulnerability. This fits our definition of vulnerability – if a possible poverty episode becomes less severe, vulnerability decreases.

Use (R2) and rewrite (A4) as follows: for any $(z,\mathbf{y},\mathbf{p})$ and $p_1\geq e>0$,

$$\sum_{i=1}^{n}p_i\upsilon(z,\tilde{y}_i)\begin{Bmatrix}\geq\\ \leq\end{Bmatrix}(p_1-e)\upsilon(z,\tilde{y}_1)+(p_2+e)\upsilon(z,\tilde{y}_2)+\sum_{i=3}^{n}p_i\upsilon(z,\tilde{y}_i)\ \text{iif}\ \tilde{y}_1\begin{Bmatrix}\leq\\ \geq\end{Bmatrix}\tilde{y}_2.$$

Take $\tilde{y}_1<\tilde{y}_2$, then

$$p_1\upsilon(z,\tilde{y}_1)+p_2\upsilon(z,\tilde{y}_2)>(p_1-e)\upsilon(z,\tilde{y}_1)+(p_2+e)\upsilon(z,\tilde{y}_2)$$

which implies

$$\upsilon(z,\tilde{y}_1)>\upsilon(z,\tilde{y}_2)\ \text{for}\ \tilde{y}_1<\tilde{y}_2 \qquad \text{(R3).}$$

Axioms 1 to 4 have sufficed so far. The next step imposes convexity as a further condition on function υ by invoking Axiom 5. In fact, it follows directly from (R2) and (A5) that

$$\sum_{i=1}^{n} p_i \upsilon(z, \tilde{y}_i) > \sum_{i=1}^{n} p_i \upsilon(z, \hat{y}) = \upsilon(z, \sum_{i=1}^{n} p_i \tilde{y}_i) \qquad \text{(R4).}$$

Finally, the proof is completed by Axiom 6. From (R3) and (R4) and (A6), setting λ=1/z, we have

$$V(z, \mathbf{y}, \mathbf{p}) = V\left(\frac{1}{z}z, \frac{1}{z}\mathbf{y}, \mathbf{p}\right) = \sum_{i=1}^{n} p_i \upsilon\left(1, \frac{\tilde{y}_i}{z}\right) = \sum_{i=1}^{n} p_i v(x_i), \text{ where } x_i = \frac{\tilde{y}_i}{z},$$

where v(.) is a monotonically decreasing and convex function. ■

Proof of Theorem 2

Proof. Given Theorem 1, the definition of y^C can be rearranged as

$$\sum_{i=1}^{n} p_i v(x_i) = \sum_{i=1}^{n} p_i v(x^C) = v(x^C), \text{ where } x^C = y^C/z.$$

Combining this definition with (A8),

$$\sum_{i=1}^{n} p_i v(x_i) = v\left(\frac{v^{-1}\left(\sum_{i=1}^{n} p_i v(\kappa x_i)\right)}{\kappa}\right)$$

which can be rewritten as

$$\sum_{i=1}^{n} p_i g(h_i) = g\left(\sum_{i=1}^{n} p_i h_i\right), \text{ where } h_i = v(\kappa x_i) \text{ and } g(h_i) = v\left(\frac{v^{-1}(h_i)}{\kappa}\right).$$

From Jensen's inequalities, any concave (convex) segment in function g would turn this equality into a lower-than (greater-than) inequality. Hence, since function v is continuous, it must be the case that

$$\frac{d^2 g}{dh_i^2} = \frac{v'(x_i)}{\left[\kappa v'(\kappa x_i)\right]^2 x_i}\left[\frac{v''(x_i)}{v'(x_i)} x_i - \frac{v''(\kappa x_i)}{v'(\kappa x_i)} \kappa x_i\right] = 0$$

As the difference in brackets must be zero and κ can take any positive value, it follows that

$$\left(\frac{d\ln(v'(x_i))}{dx_i} = \right)\frac{v''(x_i)}{v'(x_i)} = \frac{\alpha - 1}{x_i}, \text{ where } \alpha \text{ is a constant.}$$

Integrating and exponentiating,

$$v'(x_i) = e^{(\alpha-1)\ln(x_i)+\ln(\beta_1)} = \beta_1 x_i^{\alpha-1}$$, where β_1 is a constant.

Integrating again,

$$v(x_i) = \beta_0 + \beta_1 \frac{x_i^{\alpha}}{\alpha}.$$

In order to satisfy monotonicity (R3) and convexity (R4), $\beta_1<0$ and $\alpha<1$. Note when $\alpha \to 0$, $v(x_i) \to \beta_0+\beta_1 \ln(x_i)$.

Going back to Theorem 1, the corresponding vulnerability measure is

$$V(z,\mathbf{y},\mathbf{p}) = \sum_{i=1}^{n} p_i \left(\beta_0 + \beta_1 \frac{x_i^{\alpha}}{\alpha} \right) = \beta_0 + \beta_1 \sum_{i=1}^{n} p_i \frac{x_i^{\alpha}}{\alpha} \qquad \text{(R5).}$$

We now turn to Axiom 7 – from (A2) and given monotonicity (R3), it is apparent that

$$\mathbf{y}^{max} = \arg\max_y \{V(z,\mathbf{y},\mathbf{p})\} = (0,0,...,0);$$
$$\mathbf{y}^{min} = \arg\min_y \{V(z,\mathbf{y},\mathbf{p})\}, \text{ with } \tilde{\mathbf{y}}^{min} = (z,z,...,z).$$

Substituting these vectors into (R5) and normalising as imposed by Axiom 7,

$$V(z,(0,0,...,0),\mathbf{p}) = 1 \Rightarrow \beta_0 + \beta_1 \sum_{i=1}^{n} p_i(0) = \beta_0 = 1, \text{ for } \alpha \neq 0$$
$$V(z,(z,z,...,z),\mathbf{p}) = 0 \Rightarrow \beta_0 + \beta_1 \sum_{i=1}^{n} p_i \frac{1}{\alpha} = \beta_0 + \frac{\beta_1}{\alpha} = 0 \Rightarrow \frac{\beta_1}{\alpha} = -1$$

As $\beta_1<0$, we must impose $0<\alpha<1$. We can finally write our vulnerability measure as

$$V^*_{\alpha} = 1 - \sum_{i=1}^{n} p_i x_i^{\alpha}, \text{ where } 0<\alpha<1. \qquad \blacksquare$$

Proof of Theorem 3

Proof. Again, we can write

$$\sum_{i=1}^{n} p_i v(x_i) = \sum_{i=1}^{n} p_i v(x^C) = v(x^C), \text{ where } x^C = y^C/z.$$

Combining this definition with (A9),

$$\sum_{i=1}^{n} p_i v(x_i) = v\left(v^{-1}\left(\sum_{i=1}^{n} p_i v(x_i+\tau) \right) - \tau \right)$$

From this point onwards, the exercise mirrors the proof of Theorem 2, provided the following redefinitions apply: $\kappa=1+\tau/x_i$, and $g(h_i)=v(v^{-1}(h_i)-\tau)$. ■

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